Cambridge International AS & A Level

FURTHER MATHEMATICS **9231/13**

Paper 1 Further Pure Mathematics 1 **May/June 2021**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2021 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:
Marks must be awarded in line with:
- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:
Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:
Marks must be awarded **positively**:
- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:
Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:
Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:
Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1(a)	$\frac{\sin(r+1)}{\cos(r+1)}-\frac{\sin r}{\cos r}=\frac{\sin(r+1)\cos r-\cos(r+1)\sin r}{\cos(r+1)\cos r}=\frac{\sin(r+1-r)}{\cos(r+1)\cos r}$ OR $\frac{\sin(r+1)}{\cos(r+1)}-\frac{\sin r}{\cos r}=\frac{\sin r\cos 1+\cos r\sin 1}{\cos(r+1)}-\frac{\sin r}{\cos r}$ $=\frac{\sin r\cos r\cos 1+\cos^2 r\sin 1-\sin r\cos r\cos 1+\sin^2 r\sin 1}{\cos(r+1)\cos r}$	**M1**	Applies all relevant correct addition formulae from MF19 for the route chosen and combines into a single fraction.
	$=\frac{\sin 1}{\cos(r+1)\cos r}$ $\frac{\tan r+\tan 1}{1-\tan r\tan 1}-\tan r=\frac{\tan r+\tan 1-\tan r+\tan^2 r\tan 1}{1-\tan r\tan 1}$ $=\frac{\tan 1\sec^2 r}{1-\tan r\tan 1}=\frac{\tan 1}{\cos r(\cos r-\sin r\tan 1)}=\frac{\sin 1}{\cos r(\cos r\cos 1-\sin r\sin 1)}$ $=\frac{\sin 1}{\cos r\cos(r+1)}$	**A1**	SC B2 for this route completely correct to AG.
		2	
1(b)	$[\sin 1]\sum_{r=1}^{n}u_r=\tan 2-\tan 1+\tan 3-\tan 2+\ldots+\tan(n+1)-\tan n$	**M1 A1**	Shows enough terms for cancellation to be clear. There must be three complete terms including first and last. Cancelling may be implied.
	$\sum_{r=1}^{n}u_r=\frac{\tan(n+1)-\tan 1}{\sin 1}$	**A1**	OE and ISW.
		3	

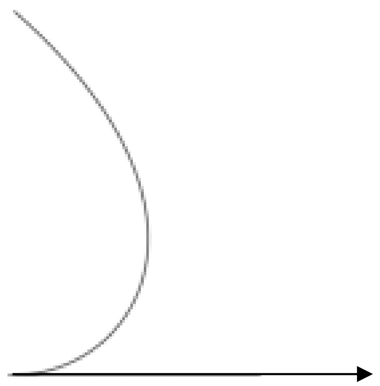
Question	Answer	Marks	Guidance
1(c)	$\tan(n+1)$ oscillates as $n \to \infty$ so $u_1 + u_2 + u_3 + \ldots$ does not converge.	**B1**	States 'oscillates' or refers to diverging values of $\tan(n+1)$, or states that $\tan(n+1)$ does not tend to a limit.
		1	
2(a)	$S_1 = 2$	**B1**	
	$S_2 = S_1^{\ 2} - 2(0)$	**M1**	Uses formula for sum of squares.
	$= 4$	**A1**	Correct answer implies M1A1.
		3	
2(b)(i)	$S_{n+3} = 2S_{n+2} - \frac{3}{2}S_n$	**B1**	CAO or as a single fraction.
		1	
2(b)(ii)	$S_4 = 2S_3 - \frac{3}{2}S_1 = 2(2S_2 - \frac{3}{2}S_0) - \frac{3}{2}S_1$	**M1**	Uses their recursive formula from part **(i)** to find S_4 $\left[S_3 = \frac{7}{2}\right]$.
	$= 4$	**A1**	
		2	
2(c)	$x = 2 - y$	**B1**	SOI
	$2(2-y)^3 - 4(2-y)^2 + 3 = 0$	**M1**	Makes *their* substitution.
	$2y^3 - 8y^2 + 8y - 3 = 0$	**A1**	OE but must be an equation.
		3	

Question	Answer	Marks	Guidance
2(d)	$\frac{\frac{8}{2}}{\frac{-3}{2}}$ OR Or use $2S_2 - 8S_1 + 8S_0 - 3S_{-1} = 0$ with substitution of their values	**M1**	Uses $\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} = \frac{\alpha'\beta' + \beta'\gamma' + \gamma'\alpha'}{\alpha'\beta'\gamma'}$.
	$= \frac{8}{3}$	**A1 FT**	FT from **2(c).**
		2	

Question	Answer	Marks	Guidance
3(a)	$5\times1^4+1^2=\frac{1}{2}(2)^2(2+1)[=6]$ so H_1 is true.	**B1**	Checks base case.
	Assume that $\sum_{r=1}^{k}\left[\left(5r^4+r^2\right)\right]=\frac{1}{2}k^2(k+1)^2(2k+1)$	**B1**	States inductive hypothesis [for some *k*] including the algebraic form. If says for **ALL** *k*, then B0.
	$\sum_{r=1}^{k+1}\left[\left(5r^4+r^2\right)\right]=\frac{1}{2}k^2(k+1)^2(2k+1)+5(k+1)^4+(k+1)^2$	**M1**	Considers sum to $k+1$.
	$\frac{1}{2}(k+1)^2\left(2k^3+k^2+10(k+1)^2+2\right)$	**M1**	Take out the factor of $(k+1)^2$ OR expands the summation expression and the target expression for $k+1$ and collects like terms for both.
	$\frac{1}{2}(k+1)^2(2k^3+11k^2+20k+12)=\frac{1}{2}(k+1)^2(k+2)^2(2k+3)$	**A1**	Factorises or having expanded, checks explicitly. At least one intermediate step seen following the award of M1 before reaching the answer.
	So H_{k+1} is true. By induction, H_n is true for all positive integers *n*.	**A1**	States conclusion. Implication must be clearly expressed.
		6	
3(b)	$5\sum_{r=1}^{n}r^4+\frac{1}{6}n(n+1)(2n+1)=\frac{1}{2}n^2(n+1)^2(2n+1)$	**M1**	Uses correct formula for $\sum r^2$.
	$[5]\sum_{r=1}^{n}r^4=\frac{1}{6}n(n+1)(2n+1)(3n(n+1)-1)$	**M1**	Makes $\sum r^4$ the subject and takes out all linear factors and the remaining term is of correct form.
	$\sum_{r=1}^{n}r^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$	**A1**	CAO
		3	

Question	Answer	Marks	Guidance
4(a)	$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2+k & k \\ 8 & -1 \\ 2 & 0 \end{pmatrix}$ Or $\begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 4 \\ 8 & -k-2 & -k+6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$	**M1 A1**	Multiplies two matrices correctly.
	$\begin{pmatrix} 10 & -1 \\ -k+14 & -k-2 \end{pmatrix}$	**A1**	
		3	
4(b)	$2\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} - k\begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 5 & -1 \\ 1 & 0 \end{vmatrix} = 0$ leading to $-2-2k+k=0$	**M1 A1**	Sets determinant equal to zero and forms linear equation.
	$k=-2$	**A1**	
		3	

Question	Answer	Marks	Guidance
4(c)	$\begin{pmatrix} 10 & -1 \\ 16 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10x - y \\ 16x \end{pmatrix}$	**M1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$. Allow $q\begin{pmatrix} 1 \\ m \end{pmatrix}$ where q is x, t or a nonzero number.
	$10x - mx = X$ and $16x = mX$	**M1 A1**	Uses $y = mx$ and $Y = mX$. Expect $16x = m(10x - mx)$.
	$16 = 10m - m^2$ $[m^2 - 10m + 16 = 0]$	**A1**	OE
	$y = 2x$ and $y = 8x$	**A1**	
		5	

Question	Answer	Marks	Guidance
5(a)	 [O] [$\theta = 0$]	**B1**	Approximately correct curve passing through the pole, O, with correct domain.
		B1	r strictly increasing over the domain 0 to $\frac{\pi}{2}$.
		B1	Correct form at O. Approx tangential to initial line at O.
		3	

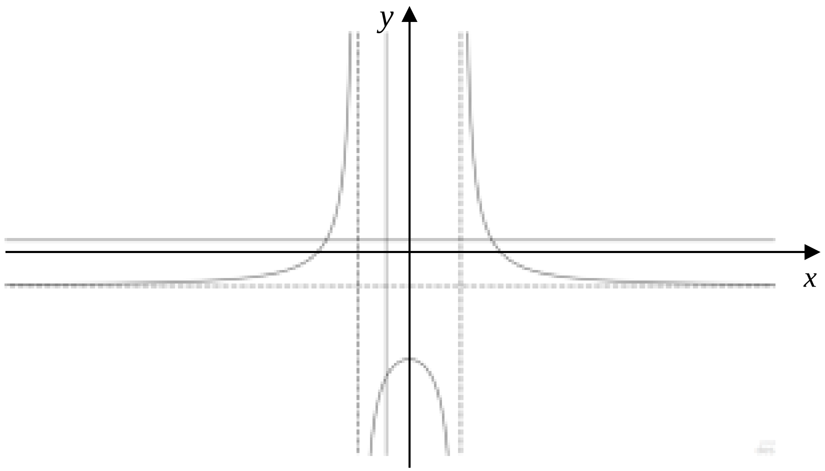
Question	Answer	Marks	Guidance
5(b)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\left(\frac{1}{\pi-\theta}-\frac{1}{\pi}\right)^2 \mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2\ \mathrm{d}\theta$ with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\frac{1}{(\pi-\theta)^2}-\frac{2}{\pi(\pi-\theta)}+\frac{1}{\pi^2}\,\mathrm{d}\theta$	**M1**	Expands.
	$\frac{1}{2}\left[\frac{1}{\pi-\theta}+\frac{2}{\pi}\ln(\pi-\theta)+\frac{\theta}{\pi^2}\right]_0^{\frac{1}{2}\pi}$	**M1 A1**	Integrates all terms to obtain correct form.
	$\frac{1}{2}\left(\frac{2}{\pi}+\frac{2}{\pi}\ln\frac{\pi}{2}+\frac{1}{2\pi}-\left(\frac{1}{\pi}+\frac{2}{\pi}\ln\pi\right)\right)=\frac{1}{2}\left(\frac{3}{2\pi}+\frac{2}{\pi}\ln\frac{1}{2}\right)$	**M1**	Substitute limits into an expression of the correct form and simplify the log terms.
	$=\frac{3-4\ln 2}{4\pi}$	**A1**	AG
		6	

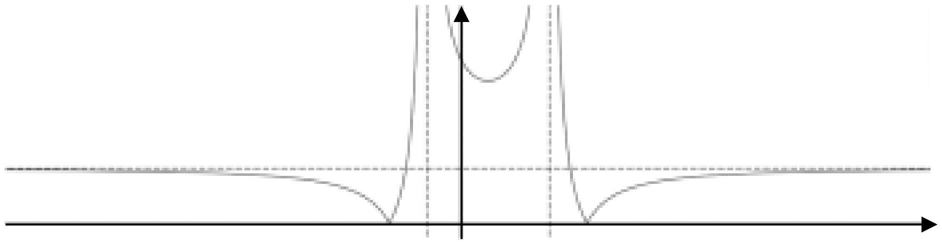
Question	Answer	Marks	Guidance
6(a)	$-(-2\mathbf{i}-2\mathbf{j}+4\mathbf{k})+(-\mathbf{i}-2\mathbf{j}+\mathbf{k})=\mathbf{i}-3\mathbf{k}$	**B1**	OE. Finds direction vector from P to a point of l_1.
	$\mathbf{r}=-2\mathbf{i}-2\mathbf{j}+4\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j})+\mu(\mathbf{i}-3\mathbf{k})$ $\boldsymbol{r}=-\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}+\lambda(2i-3j)+\mu(\boldsymbol{i}-3\boldsymbol{k})$	**B1 FT**	OE. FT *their* **i** – 3**k**
		2	

Question	Answer	Marks	Guidance
6(b)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 3 & -1 & 3 \end{vmatrix} = \begin{pmatrix} -9 \\ -6 \\ 7 \end{pmatrix}$	**M1 A1**	OE. Finds vector perpendicular to Π_2.
	$-9(3)-6(0)+7(-2)=-41$	**M1**	Uses point on Π_2.
	$9x+6y-7z=41$	**A1**	
		4	
6(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 1 & 0 & -3 \end{vmatrix} = \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix} \sim \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$	**M1 A1**	Finds vector perpendicular to Π_1.
	$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 6 \\ -7 \end{pmatrix} = \sqrt{14}\sqrt{166}\cos\alpha$ leading to $\cos\alpha = \dfrac{32}{\sqrt{14}\sqrt{166}}$	**DM1** **A1 FT**	Dot product using their normal vectors.
	48.4°	**A1**	0.845 rad
		5	

Question	Answer	Marks	Guidance
6(d)	$\overrightarrow{OF} = \overrightarrow{OQ} + \overrightarrow{QF} = -5\begin{pmatrix}-2\\-2\\4\end{pmatrix} + t\begin{pmatrix}9\\6\\-7\end{pmatrix} = \begin{pmatrix}10+9t\\10+6t\\-20-7t\end{pmatrix}$	**M1 A1 FT**	Scales $\overline{OP}$ by a factor of ±5 and uses multiple of their normal to Π_2.
	$9(10+9t)+6(10+6t)-7(-20-7t)=41$ leading to $290+166t=41$	**DM1**	Substitutes into the equation of Π_2.
	$t=-\frac{3}{2}$ leading to $\overrightarrow{OF} = \begin{pmatrix}-\frac{7}{2}\\1\\-\frac{19}{2}\end{pmatrix}$	**A1**	
		4	

Question	Answer	Marks	Guidance
7(a)	$x=\dfrac{1-\sqrt{5}}{2},\ x=\dfrac{1+\sqrt{5}}{2}$	**B1**	Vertical asymptotes. Must be exact.
	$y=-1$	**B1**	Horizontal asymptote.
		2	
7(b)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(1+x-x^2)(2x-1)-(x^2-x-3)(1-2x)}{\left(1+x-x^2\right)^2}$	**M1**	Finds $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.
	$(2x-1)(-2)=0$	**M1**	Sets their $\dfrac{dy}{dx}$ equal to 0 and forms equation.
	$\left(\frac{1}{2},-\frac{13}{5}\right)$	**A1**	WWW
		3	

Question	Answer	Marks	Guidance
7(c)		B1	Both axes labelled and correct asymptotes shown.
		B1	Correct shape and position, with all asymptotic behaviour clear.
	$\left(\frac{1}{2}+\frac{1}{2}\sqrt{13},0\right)$, $\left(\frac{1}{2}-\frac{1}{2}\sqrt{13},0\right)$, $(0,-3)$	B1	States exact coordinates of intersections with axes.
		3	

Question	Answer	Marks	Guidance
7(d)		B1 FT	FT from sketch in **(c).**
		B1	Correct shape as x tends to infinity and intersections with x axis.
	$\dfrac{x^2-x-3}{1+x-x^2}=3$ or $\dfrac{x^2-x-3}{1+x-x^2}=-3$ $4x^2-4x-6=0$ or $-2x^2+2x=0$	M2	Finds critical points, award M1 for each case.
	$x=\frac{1}{2}+\frac{1}{2}\sqrt{7}$, $x=\frac{1}{2}-\frac{1}{2}\sqrt{7}$, $x=0$, $x=1$	A1	Must be exact.
	$x<\frac{1}{2}-\frac{1}{2}\sqrt{7}$, $0<x<1$, $x>\frac{1}{2}+\frac{1}{2}\sqrt{7}$	A1 FT	Must be three distinct regions and strict inequalities.
		6	